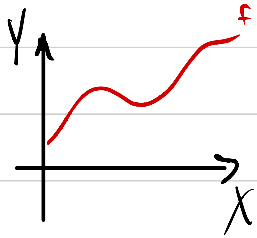
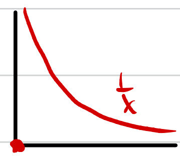
Descriptive Set Theory
Lecture 20

Obs. If a function $f : X \to Y$, X, Y top. spaces, Hausdorf, is continuous, then its graph $G_f \subseteq X \times Y$ is closed. In fact, if f is continuous $X \to (Y, \mathcal{T}')$, where $\mathcal{T}'$ is a coarser Hausd. top on Y, then G_f is still closed in $X \times Y$ with the finer top on Y.



Converse is false.



$$f : [0,\infty) \to [0,\infty)$$
$$x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{o.w.} \end{cases}$$

Top. black magic (Ruiyuan (Ronnie) Chen). Let X, Y be Polish, with top. $\mathcal{T}_X$ and $\mathcal{T}_Y$ resp. Let $f : X \to Y$ be a function with closed graph. If $\mathcal{T}_X'$ is the coarsest refinement of $\mathcal{T}_X$ making f continuous, i.e. $\mathcal{T}_X' := \mathcal{T}_X + f^{-1}(\mathcal{T}_Y)$, then $\mathcal{T}_X'$ is still Polish.

Proof. By the continuity of $f : (X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y)$, $\mathcal{T}_X'$ is homeomorphic the the top $\mathcal{T}_X' \times \mathcal{T}_Y$ restricted to the graph G_f. But this top is the same on G_f as $\mathcal{T}_X \times \mathcal{T}_Y$ (by black magic) so $\mathcal{T}_X' \times \mathcal{T}_Y |_{G_f}$ is Polish being a closed subset of the Polish top $\mathcal{T}_X \times \mathcal{T}_Y$ on $X \times Y$. $\square$

Remark. The way we will use this is as follows:

- Start with a cont. $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$, $\mathcal{T}_X, \mathcal{T}_Y$ Polish.
- Take any Polish refinement $\mathcal{T}_Y'$ of $\mathcal{T}_Y$.
- G_f still stays closed in $\mathcal{T}_X \times \mathcal{T}_Y'$.
- Then $\mathcal{T}_X' := \mathcal{T}_X + f^{-1}(\mathcal{T}_Y')$ is still Polish and $f: (X, \mathcal{T}_X') \to (Y, \mathcal{T}_Y')$ is still continuous.

Analytic sets.

What happens when we project a Borel set, i.e. say X, Y are Polish, $B \subseteq X \times Y$ Borel, is $\mathrm{proj}_X B$ Borel? Lebesgue showed yes but Souslin showed no, which gave rise to a new class of sets:

Def. A subset A of a Polish space X is called *analytic* if it's a continuous image of a Borel set, i.e. $\exists$ Polish Y and $B \subseteq Y$ and $f: Y \to X$ continuous s.t. $A = f(B)$.

Prop. For a Polish X and $A \subseteq X$, TFAE:

(1) $\exists$ Polish Y and a Borel $B \subseteq X \times Y$ s.t. $A = \mathrm{proj}_X(B)$.

(2) A is analytic, i.e. $\exists$ Polish Z and Borel $B \subseteq Z$ and continuous $f : Z \to X$ s.t. $A = f(B)$.

(3) $\exists$ Polish Z, Borel $B \subseteq Z$ and a Borel $f : Z \to X$ s.t. $A = f(B)$.

(4) $\exists$ Polish Z and a continuous $f : Z \to X$ s.t. $A = f(Z)$.

(5) $\exists$ continuous $f : \mathbb{N}^{\mathbb{N}} \to X$ s.t. $A = f(\mathbb{N}^{\mathbb{N}})$.

(6) $\exists$ closed $C \subseteq X \times \mathbb{N}^{\mathbb{N}}$ s.t. $A = \mathrm{proj}_X(C)$.

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3) is trivial. (3)$\Rightarrow$(4). Make f continuous and $B \subseteq Z$ clopen by refining the Polish top. on Z into a finer Polish top, so $f : B \to X$ is continuous and $A = f(B)$. (4)$\Rightarrow$(5) is due to the fact that every Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$. (5)$\Rightarrow$(6) let $C := G_f$ the graph of f. $\square$

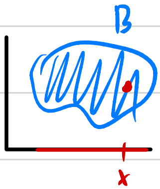
Let $\Sigma^1_1(X)$ be the collection of analytic subsets of a Polish space X, and let $\Pi^1_1(X) := \neg\Sigma^1_1(X)$ be the class of co-analytic sets. Let $\Delta^1_1(X) := \Sigma^1_1(X) \cap \Pi^1_1(X)$. Note that $\mathcal{B}(X) \subseteq \Delta^1_1(X)$.

For a Polish space Y and a class Γ of subsets of Polish spaces (say $\Gamma = \mathcal{B}$ or $\Gamma = \Pi^0_1$), define, for each Polish X,

$$\exists^Y \Gamma(X) := \{\mathrm{proj}_X(B) : B \in \Gamma(X \times Y)\}.$$

Why this notation? If $A = \mathrm{proj}_X(B)$, $B \subseteq X \times Y$, $\forall x \in X$

$$x \in A \iff \exists y \in Y \ (x,y) \in B$$



Note $\Sigma^1_1(X) = \exists^{\mathbb{N}^\mathbb{N}} \Pi^0_1(X)$, by (6) of Propos.

Closure properties of Σ^1_1. Σ^1_1 is closed under

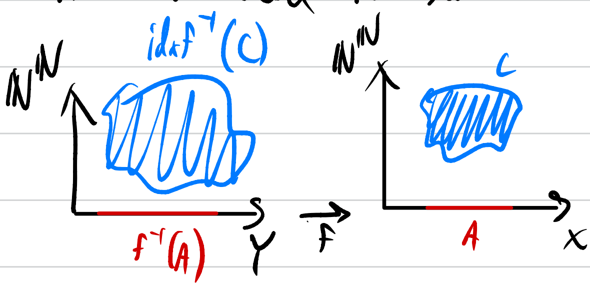
(a) continuous images and preimages

(b) ctbl unions and ctbl intersections

(a') Borel images and preimages.

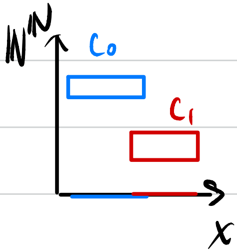
Proof. The closure under cont. images is trivial.

For preimages, let $A \subseteq X$ be the proj. of a closed $C \subseteq X \times \mathbb{N}^\mathbb{N}$ and let $f: Y \to X$ be continuous. Need to show that $f^{-1}(A)$ is still analytic



$$id \times f : Y \times \mathbb{N}^\mathbb{N} \to X \times \mathbb{N}^\mathbb{N}$$
$$(y,z) \mapsto (f(y), z).$$

For <u>intersections</u>, let $A_n \subseteq X$ be analytic, so $A_n = \mathrm{proj}_X C_n$ where $C_n \subseteq X \times \mathbb{N}^\mathbb{N}$ is closed. For $x \in X$,



$$
\begin{aligned}
x \in \bigcap_n A_n &\iff \forall n \ x \in A_n \\
&\iff \forall n \ \exists y_n \in \mathbb{N}^\mathbb{N} \ (x, y_n) \in C_n \\
&\iff \underbrace{\exists (y_n) \in (\mathbb{N}^\mathbb{N})^\mathbb{N} \ \underbrace{\forall n \ \underbrace{(x, y_n) \in C_n}_{\text{closed}}}_{\text{closed}}}_{\Sigma^1_1}
\end{aligned}
$$

We leave unions as an exercise. $\square$

<u>Thm</u> (Souslin). Every unctbl Polish space X has an X-universal set for $\Sigma^1_1(X)$, i.e. $\exists\, \mathcal{U} \subseteq X \times X$ analytic s.t. $\{\mathcal{U}_x : x \in X\} = \Sigma^1_1(X)$.

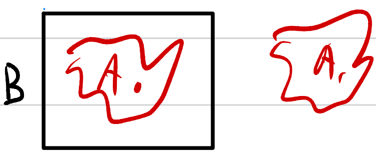
Proof. Recall that $\Sigma^1_1(X) = \exists^{\mathbb{N}^\mathbb{N}} \Pi^0_1(X \times \mathbb{N}^\mathbb{N})$.
Let $\mathcal{U} \subseteq X_0 \times X_1 \times \mathbb{N}^\mathbb{N}$ be an X-universal set for $\Pi^0_1(X \times \mathbb{N}^\mathbb{N})$, where $X_0 = X_1 = X$. Then the set $\mathcal{U}' \subseteq X_0 \times \mathbb{N}^\mathbb{N}$ defined by $\mathcal{U}' := \mathrm{proj}_{X_0 \times X_1} \mathcal{U}$ is as desired. $\square$

<u>Cor</u> (Souslin). $\Delta^1_1(X) \subsetneq \Sigma^1_1(X),\ \Pi^1_1(X)$, for any unctbl Polish X.

Proof. Let $U \subseteq X \times X$ be a universal set for $\Sigma^1_1(X)$. Then $\mathrm{AntiDiag}(U) := \{x \in X : (x,x) \in U^c\}$ is coanalytic being the preimage of U^c of $x \mapsto (x,x)$, and it is not analytic since it's not $= U_x$ for any $x \in X$, by Cantor. $\square$

Note. For a σ-compact space X, like $X := \mathbb{R}$, if $C \subseteq X \times X$ is closed, then $\mathrm{proj}_X C$ is still σ-compact hence F_σ.

Analytic Separation (Luzin). If $A_0, A_1 \subseteq X$ Polish are disjoint analytic sets, then $\exists\, B \subseteq X$ Borel separating them, i.e. $B \supseteq A_0$ and $B^c \supseteq A_1$.



Cor (Souslin). $\mathcal{B}(X) = \Delta^1_1(X)$ for any Polish X.

Proof. If $A \subseteq X$ is analytic and A^c is also analytic, then by separation, $\exists$ Borel $B \subseteq X$ s.t.

$$B \supseteq A \text{ and } B^c \supseteq A^c \implies A = B.$$

$\square$